SOME TOPOLOGICAL ASPECTS OF GENERALIZED CONE METRIC SPACES

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Abstract

In these paper we consider the p-quasi cone metric space and p- cone metric space as topological spaces. We define the topology in p-quasi cone metric space. We study some properties related with open and closed sets, the convergent and Cauchy sequences. Also we give some theorems related with completeness and compactness of p – quasi cone metric space. Our results are generalizations of many theorems in metric space, cone metric space and quasi cone metric space.

Keywords*: p-quasi cone metric space, topology, completeness, compactness, convergence*

1. Introduction

In 2007, Huang and Zhang [4] introduced the concept of cone metric space replacing the set of real numbers with an ordered Banach vector space. They studied fixed points in these spaces. There are many authors as Rezapour, Abdeljawad, who have worked with these ideas. Sh. Rezapour(Rezapour.Sh, Derafshpour.M, Hamlbarani R., 2008) and P. D. Proinov(Proinov.P. 2013) have defined the topology in cone metric space. Because quasimetric space is more general than metric space and is a subject of intensive research in the context of topology and theoretical computer science, Abdeljawad and Karapinar (Abdeljawad. T, Karapinar. E. 2009) and Sonmez have given a definition of quasi-cone metric space which extends the quasi-metric space.

In 2014, E. Sila et.al (Sila. E, Hoxha. E, Dule. K, 2014) introduced the concept of p – quasi cone metric space which generalize the concept of quasi cone metric, cone metric space and p-cone metric space. Until now we have studied some fixed point results in p –quasi cone metric space.

Now we recall some known notions, definitions and results which are used in this paper.

2. Preliminaries

Definition1.1[4] Let$E$ be a real Banach space and P be a subset of E. P is called a cone if and only if

(i) P is closed, $P \neq \Phi$, $P \neq \{0\}$;

(ii) for all x, $y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in R^{+}$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, int$P$ denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$, for all $x, y \in E$. The least positive k satisfying this is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent; that is if x_n is a sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$, for some $y \in E$, then there is $x \in E$ such that

$\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every sequence which is bounded below is convergent.

Definition 1.2[4] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x=y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3 (Abdeljawad. T, Karapinar. E. 2009) Let X be a nonempty set. Suppose the mapping $q : X \times X \to E$ satisfies

(i) $0 \leq q(x, y)$ for all $x, y \in X$,

(ii) $q(x, y) = 0$ if and only if $x = y$;

(iii) $q(x,z) \leq q(x,y) + q(y,z)$ for all $x, y, z \in X$

then q is called a quasi- cone metric on X, and (X, q) is called a quasi-cone metric space.

Now, we state our definition which is more general than quasi-cone metric space.

Definition 1.4 (Sila. E, Hoxha. E, Dule. K, 2014) Let X be a nonempty set and $p \geq 1$. Suppose the mapping $q_p : X \times X \to E$ satisfies

(i) $0 \leq q_p(x, y)$ for all $x, y \in X$,

(ii) $q_p(x, y) = 0$ if and only if $x = y$;

(iii) $q_p(x,z) \leq p(q_p(x,y) + q_p(y,z))$ or all $x, y, z \in X$

Then q_p is called a p-quasi -cone metric on X, and (X, q_p) is called a p-quasi -cone metric space.

Example1.5 Let $X = (0,\infty), E = R^2, P = \{(x,y) \in E, x, y \in R^+\}$ and $q_2 : X \times X \to E$ defined by

$$q_2(x,y) = \begin{cases} ((x-y)^2, \alpha(x-y)^2),\ x > y \\ (0,0),\ x < y \end{cases}, \text{ where } \alpha \in R^+.$$

2.Main Results

2.1 The topology in p-quasi cone metric space

Sh. Rezapour (Rezapour.Sh, Derafshpour.M, Hamlbarani R., 2008) and P. D. Proinov (Proinov.P. 2013) have defined topology in cone metric space. Now we define topology in p-quasi cone metric space and we give some topological properties of these spaces.

Let E be an ordered vector Banach space, $P \subset E$ a regular and normal cone in E with constant of normality $K \geq 1$ and (X, q_p) a p-quasi cone metric space. We take $a \in X$ and $c \gg 0$, $c \in P$.

Definition2.1.1 *Open right ball of radius c>>0, centered at a is called the set* $B^d(a,c) = \{x \in X : q_p(x,a) \ll c\}$.

Definition 2.1.2 *Open left ball of radius c>>0, centered at a is called the set* $B^m(a,c) = \{x \in X : q_p(a,x) \ll c\}$.

Definition 2.1.3 *Closed right ball of radius c>>0, centered at a is called the set* $B^d(a,c) = \{x \in X : q_p(x,a) \leq c\}$. *Closed left ball of radius c>>0, centered at a is called the set* $B^m(a,c) = \{x \in X : q_p(a,x) \leq c\}$.

Theorem 2.1.4 Let (X, q_p) be a *p*-quasi cone metric space. The family defined as follow $\tau_{q_p}^d = \{\phi, X, G \subset X, \forall a \in G, \exists\, B^d(a,x) \subset G\}$ is topology in X.

Proof: The family $\tau_{q_p}^d$ satisfies the conditions of being a topology because:

1) $X, \phi \in \tau_{q_p}^d$.
2) For every $(G_i)_{i\in I} \subset \tau_{q_p}^d$ and $a \in \bigcup_{i\in I} G_i$ there exist an open left ball $B_{i_0}^d(a, c_{i_0}) \subset G_{i_0}$. From $B_{i_0}^d(a, c_{i_0}) \subset G_{i_0} \subset \bigcup_{i\in I} G_i$ we have $\bigcup_{i\in I} G_i \in \tau_{q_p}^d$.
3) For every G_1 and G_2 from $\tau_{q_p}^d$ and $a \in G_1 \cap G_2$ there exist $B_1^d(a, c_1) \subset G_1$ and $B_2^d(a, c_2) \subset G_2$. For $c_1 \gg 0$ and $c_2 \gg 0$ from Theorem 1.1.7 there exist $e \gg 0$, such that $e \ll c_1$ and $e \ll c_2$ and $B^d(a,e) \subset B_1^d(a,c_1) \cap B_2^d(a,c_2) \subset G_1 \cap G_2$. So we have $G_1 \cap G_2 \in \tau_{q_p}^d$.

The topology $\tau_{q_p}^d$ is called right topology obtained from *p*-quasi cone metric q_p. In the same manner we define the left topology $\tau_{q_p}^m$ obtained from p-quasi cone metric q_p.

The following propositions are given for right topology $\tau_{q_p}^d$. In the same manner we can formulate these propositions for left topology $\tau_{q_p}^m$ in (X, q_p).

Definition 2.1.5 The set $A \subset X$ is called right opened if $A \in \tau_{q_p}^d$.

Definition 2.1.6 The set $V \subset X$ is right neighborhood of point *a* if there exist an opened right ball centered at *a* that is contained in V ($B^d(a,c) \subset V$).

Proposition 2.1.7 The topology $\tau_{q_p}^d$ in (X, q_p) satisfies the first axiom of countability.

Proof: The family $S_d(a) = \{B^d(a,c) : c \gg 0\}$ is neighborhood system base of point *a*. For a fixed element $b \gg 0$ and every $c \gg 0$, there exist $n \in N$ such that $0 \ll \frac{1}{n} b \ll c$. So $B^d(a, \frac{1}{n} b) \subset B^d(a,c)$ and $S_d^{'}(a) = \{B^d(a, \frac{1}{n} b) : n \in N\}$ is a neighborhood system base of point *a* and the set $S_d^{'}(a)$ is countable. So the topology $\tau_{q_p}^d$ satisfies the first axiom of countability.

Theorem 2.1.8 The set $A \subset X$ is right opened if and only if for every point $a \in A$ there exist $B^d(a,c)$ such that $B^d(a,c) \subset A$.

This theorem is true from the definition of topology $\tau_{q_p}^d$.

Definition 2.1.9 The set $A \subset X$ is right closed if $X - A$ is right opened.

Theorem 2.1.10 The space $(X, \tau_{q_p}^d)$ is T_1.

Proof: For every $x \in X$ we have to prove that set $\{x\}$ is right closed or the set $X - \{x\}$ is right opened. Let be $a \in X - \{x\}$, then $a \neq x \Rightarrow q_p(x,a) \gg 0$. Sign as $c = q_p(x,a)$. Because $c \gg 0$ then there exist $n \in N$ such that $0 \ll \frac{c}{2n} \ll c$. Let prove that $B(a, \frac{c}{2n}) \subset X - \{x\}$. Let be $z \in B(a, \frac{c}{2n})$, then $q_p(z,a) \ll \frac{c}{2n} \ll c = q_p(x,a)$. So $z \neq x$, and $z \in X - \{x\}$. We have that $X - \{x\}$ is a right opened set and the topology $\tau^d_{q_p}$ is T_1.

Definition 2.1.11. Let (X, q_p) be a p-quasi cone metric space and (x_n) a sequence in X.

(1) The sequence (x_n) *is right convergent at* $x \in X$ if for every $c \gg 0$, there exist $n_0 \in N$ such that for every $n \geq n_0$ we have $q_p(x_n, x) \ll c$.

(2) The sequence (x_n) *is left convergent at* $x \in X$ if for every $c \gg 0$, there exist $n_0 \in N$, such that for every $n \geq n_0$ we have $q_p(x, x_n) \ll c$.

(3) The sequence (x_n) is right Cauchy in X if for every $c \gg 0$, there exist $n_0 \in N$, such that for every $n > m > n_0$, we have $q_p(x_n, x_m) \ll c$.

(4) The sequence (x_n) is left Cauchy in X if for every $c \gg 0$, there exist $n_0 \in N$, such that for every $n > m > n_0$ we have $q_p(x_m, x_n) \ll c$.

(5) Two sequences $\{x_n\}_{n \in N}$ and $\{y_n\}_{n \in N}$ are right equivalent Cauchy if for every $c \gg 0$, there exist $n_0 \in N$, such that for every $n > m > n_0$ we have $q_p(x_n, y_m) \ll c$.

(6) Two sequences $\{x_n\}_{n \in N}$ and $\{y_n\}_{n \in N}$ are left equivalent Cauchy, if for every $c \gg 0$, there exist $n_0 \in N$, such that for every $n > m > n_0$ we have $q_p(y_m, x_n) \ll c$.

Example 2.1.12 Let be $X = [0,1], E = R^2, P = \{(c_1, c_2) : c_1, c_2 \geq 0\}$. Define 2-quasi-cone metric as follow:

$$q_p(x,y) = \begin{cases} ((x-y)^2, \alpha^2 (x-y)^2), & x \geq y \\ (\alpha^2, 1), & x < y \end{cases}, \text{ where } 0 < \alpha < 1.$$

q_p is a p-quasi-cone metric in X, where $p = 2$.

Let be $x_n = \frac{1}{n}$ a sequence in $X = [0,1]$.

For $n > m$ we have $q_p(x_n, x_m) = q_p(\frac{1}{n}, \frac{1}{m}) = (\alpha^2, 1)$, which prove that this sequence is not right Cauchy.

For $n > m$, we have $q_p(x_m, x_n) = \left(\left(\frac{1}{m} - \frac{1}{n} \right)^2 ; \alpha^2 \left(\frac{1}{m} - \frac{1}{n^2} \right)^2 \right)$. So for every $c = (c_1, c_2) \in P$ such that $c >> 0, \exists n_0 \in N$ and for $n > m > n_0$ we have $\left(\frac{1}{m} - \frac{1}{n} \right)^2 < c_1$ and $\alpha^2 \left(\frac{1}{m} - \frac{1}{n^2} \right)^2 < c_2$, so $q_p(x_m, x_n) << c.$ so the sequence $(x_n = \frac{1}{n})$ is left Cauchy.

Also, in a p-quasi cone metric space X, there exist sequences that are right convergent but not left convergent in X.

2.2 The topological *p*-cone metric space

Let (X, q_p) be a *p*-quasi cone metric space, where $P \subset E$ is a minihedral cone. Define the function $d_p : X \times X \to P, \forall x, y \in X, d_p(x, y) = \max\{q_p(x, y), q_p(y, x)\}$.

The function d_p is well defined because cone P is minihedral then for every $c_1, c_2 \in P$ there exist $\max\{c_1, c_2\}$.

Theorem 2.2.1 The function $d_p(x, y)$ is *p*-cone metric in X.

Proof: $q_p(x, y) \geq 0$ and $q_p(y, x) \geq 0 \Rightarrow d_p(x, y) \geq 0$

$d_p(x, y) = 0 \Rightarrow \max\{q_p(x, y), q_p(y, x)\} = 0 \Rightarrow q_p(x, y) = q_p(y, x) = 0 \Rightarrow x = y$

$d_p(x, y) = \max\{q_p(x, y), q_p(y, x)\} = \max\{q_p(y, x), q_p(x, y)\} = d_p(y, x)$

$q_p(x, y) \leq p(q_p(x, z) + q_p(z, y))$ and $q_p(y, x) \leq p(q_p(y, z) + q_p(z, x))$

$$d_p(x, y) \leq \max\{q_p(x, y), q_p(y, x)\}$$
$$\leq p(\max\{q_p(x, z), q_p(z, x)\} + \max\{q_p(y, z), q_p(y, z)\}) = p(d_p(x, z) + d_p(z, y)).$$

This *p*-cone metric obtained by *p*-quasi cone metric q_p. The couple (X, d_p) is called *p*- cone metric space (type cone metric space).

Let define opened balls in *p*-cone metric $B(a, c) = \{x \in X : d_p(a, x) << c\}$ and closed balls $B^{'}(a, c) = \{x \in X : d_p(a, x) \leq c\}$.

In the same manner as theorem 2.1.6, we prove that the set

$$\tau_{d_p} = \{\phi, X, G \subset X, \forall a \in G, \exists B(a, x) \subset G\}$$

is topology in X .This topology is called obtained by p -cone metric d_p.

Theorem 2.2.2 Let (X, q_p) be a *p*-quasi cone metric space and $d_p(x, y) = \max\{q_p(x, y), q_p(y, x)\}$ p -cone metric obtained by p -quasi cone metric q_p, then $\tau_{d_p} = \tau_{q_p}^d \cap \tau_{q_p}^m$.

Proof: To prove this, we must see:

i) For every $c_1, c_2 >> 0$, there exist $c >> 0$ such that $B(a, c) \subset B^d(a, c_1) \cap B^m(a, c_2)$.

ii) for every $c_3 >> 0$, there exist $c_1 >> 0, c_2 >> 0$ such that $B^d(a, c_1) \cap B^m(a, c_2) \subset B(a, c_3)$.

i) For every $c_1 >> 0, c_2 >> 0$ there exist $c >> 0$ such that $c << c_1, c << c_2$ then $d_p(a, x) << c \Rightarrow \max\{q_p(a, x), q_p(x, a)\} << c \Rightarrow q_p(x, a) << c << c_1$ and $q_p(a, x) << c << c_2$, so $x \in B^d(a, c_1)$ and $x \in B^m(a, c_2) \Rightarrow x \in B^d(a, c_1) \cap B^m(a, c_2)$.

ii) for $c_3 >> 0$, we take $c_1 = c_3, c_2 = c_3$ and we have:

$x \in B^d(a, c_3) \cap B^m(a, c_3) \Rightarrow q_p(x, a) << c_3$

$q_p(a, x) << c_3 \Rightarrow \max\{q_p(x, a), q_p(a, x)\} << c_3 \Rightarrow d_p(a, x) << c_3 \Rightarrow x \in B(a, c_3)$.

The families $\beta_1 = \{B(a, c) : a \in X,\ c \in \mathring{P}\}$ and $\beta_2 = \{B^d(a, c_1) \cap B^m(a, c_2) : x \in X, c_1, c_2 \gg 0\}$ are bases for topology τ_{d_p}.

Theorem 2.2.3 The topological *p*-cone metric space τ_{d_p} is T_2.

Proof: Let be $x \neq y$, then $d_p(x,y) >> 0$. Sign as $d_p(x,y) = c >> 0.$

From theorem 1.1.7, for c □ 0 and $\frac{1}{2p}c >> 0$, $\exists n_0 \in N, c_1 = \frac{c}{2pn_0} << c.$

Then $B(x,c_1)$ and $B(y,c_1)$ are two neighborhoods of x and y respectively such that $B(x,c_1) \cap B(y,c_1) = \phi.$

If we suppose that $B(x,c_1) \cap B(y,c_1) \neq \phi$, there exist $z \in B(x,c_1) \cap B(y,c_1)$ and $d_p(x,z) << c_1$ and $d_p(z,y) << c_1$.

$c = d_p(x,y) << p(d_p(x,z) + d_p(z,y)) << p(c_1 + c_1) = p \cdot 2c_1 = \frac{2pc}{2pn_0} = \frac{c}{n_0} << c$ since $\frac{1}{n_0} << 1$.

This is a contradiction.

Definition 2.2.4. Let (X, q_p) be a p- quasi cone metric space, d_p p-cone metric obtained by q_p and (x_n) a sequence in X.

1) The sequence (x_n) *converges to x*, if it converges related to p-cone metric d_p:

$$\forall c >> 0, c \in P, \exists n_0 \in N \text{ such that } \forall n > n_0 \Rightarrow d_p(x_n, x) << c.$$

2) The sequence (x_n) is Cauchy in X if it is Cauchy related to p-cone metric d_p:

$$\forall c >> 0, c \in P, \exists n_0 \in N \text{ such that } \forall n,m > n_0 \Rightarrow d_p(x_n, x_m) << c.$$

As we see from Definition 2.2.4, Theorem 2.2.2 and Definition 2.1.11, we must claim that:

1) The sequence (x_n) converges to x if it converges left and right to x.
2) The sequence (x_n) is Cauchy in X if it is left and right Cauchy (bi-Cauchy).
3) The sequences (x_n) and (y_n) are equivalent Cauchy in X if they are left and right equivalent Cauchy.

Theorem 2.2.6 If (X, q_p) is a p-quasi cone metric space and K is constant of normality of cone P, then the following propositions are true:

i) The sequence (x_n) converges to x if and only if

$$d_p(x_n, x) \to 0 \Leftrightarrow \lim_{n\to\infty} q_p(x_n, x) = \lim_{n\to\infty} q_p(x, x_n) = 0 \Leftrightarrow \|d_p(x_n, x)\| \to 0.$$

ii) The sequence (x_n) is Cauchy if and only if

$$d_p(x_n, x_m) \to 0 \Leftrightarrow \lim_{n\to\infty} q_p(x_n, x_m) = \lim_{n\to\infty} q_p(x_m, x_n) = 0 \Leftrightarrow \|d_p(x_n, x_m)\| \to 0.$$

iii) The sequences (x_n) and (y_n) are equivalent if and only if

$$d_p(x_n, y_m) \to 0 \Leftrightarrow \lim_{n\to\infty} q_p(x_n, y_m) = \lim_{n\to\infty} q_p(y_m, x_n) = 0 \Leftrightarrow \|q(x_n, y_m)\| \to 0.$$

since $\tau^d_{q_p}, \tau^m_{q_p}$ have the first axiom of countability then τ_{d_p} has it.

Due to (X, τ_{d_p}) is T_2 and it has the first axiom of countability then it has the uniqueness of limit.

Also we can prove easily that:

Theorem 2.2.7

1) Every subsequence of a convergent sequence in p-quasi cone metric space is convergent.
2) Every convergent sequence in p-quasi cone metric space is Cauchy.

Definition 2.2.8 Let (X, q_p) be a p-quasi cone metric space and d_p - p -cone metric obtained by q_p.

1) $a \in X$ is an internal point of $A \subset X$ if there exist $B(a,c) \subset A$.
2) $a \in X$ is a meeting point of $A \subset X$, if for every $B(a,c), B(a,c) \cap A \neq \phi$.
3) $a \in X$ *is a limit point* $A \subset X$, if for every $B(a,c), B(a,c) \cap A - \{a\} \neq \phi$.
4) The set of internal points of A is called interior of A and we sign it by $\overset{o}{A}$.
5) The set of meeting points of A is called the closure of A and it is signed $\overline{A}$.

As in topological space that satisfies the first axiom of countability, it is true the following theorem:

Theorem 2.2.9

1) $a \in \overline{A}$ if and only if there exist sequence (x_n) in A, such that $x_n \to a$.
2) The subset $A \subset X$ is closed, if and only if every sequence (x_n) in A does not converge to any point of complement of A.
3) The set $A \subset X$ is closed if and only if $A = \overline{A}$.

If (X, q_p) is p-quasi cone metric space and P is minihedral cone then we can define bounded set in X and the diameter of a set A.

Definition 2.2.10

1) The set $A \subset X$, where (X, q_p) is p-quasi cone metric space, is bounded if there exist $B(a,c)$ such that $A \subset B(a,c)$.
2) *Diameter of bounded set* $A \subset X$ we call

$$\delta(A) = \sup\{q_p(x,y), q_p(y,x) : x, y \in A\} = \sup\{d_p(x,y) : x, y \in A\}$$

Theorem 2.2.11. If set $A \subset X$ is bounded then $\overline{A}$ is bounded too and $\delta\left(\overline{A}\right) \leq p^2 \delta\left(A\right)$.

Proof: We take $x, y \in \overline{A} \Rightarrow \exists (x_n) \to x, (y_n) \to y$ where (x_n) and (y_n) are in A.

Due to Theorem 1.4.7 we have that $d_p(x,y) \leq k^2 \underline{\lim}_n d_p(x_n, y_n)$ and $d_p(x,y) \leq k^2 \delta(A)$.

But $\delta(\overline{A}) = \sup\{d_p(x,y) : x, y \in \overline{A}\}$ and $k^2 \delta(A)$ is upper boundary $\{d_p(x,y) : x, y \in \overline{A}\}$. So $\delta(\overline{A}) = \sup\{d_p(x,y) : x, y \in \overline{A}\} = k^2 \delta(A)$.

2.3 Completeness and compactness in p-quasi cone metric space.

In this phragraph we give some proposition related on completeness and compactness of p-quasi cone metric space which are generalization of relevant propositions in metric space.

In (Rezapour.Sh, Derafshpour.M, Hamlbarani R., 2008) authors have generalized some propositions related compactness for cone metric space. Also in (Proinov.P. 2013) authors have given some proposition related on completeness of cone metric space. We generalize these results in p-quasi cone metric space and $d_p \to p$ -cone metric obtained by q_p.

Definition 2.3.1 The p-quasi cone metric space (X, q_p) is complete if every Cauchy sequence in it converges at a point of X.

Definition 2.3.2 The set $A \subset X$, where (X, q_p) is *p*-quasi cone metric space is complete if every Cauchy sequence in A converges to a point of A.

Theorem 2.3.3 The set $A \subset X$, where (X, q_p) is *p*-quasi cone metric space is complete if and only if it is closed.

Proof: *Necessary condition*. Let A be complete and $x \in \overline{A}$. From Theorem 1.4.9 there exist a sequence (x_n) in A such that $x_n \to x$. So the sequence (x_n) as convergent sequence is Cauchy sequence in A. Due to A is complete we have that $x_n \to y \in A$. From uniqueness of limit we have $x = y \in A$. So $A = \overline{A}$ and A is closed.

Sufficient condition. A is closed. Let (x_n) be Cauchy sequence in *A*, then it is Cauchy in *X* too. Due to X is complete, $x_n \to x \in X$. From Theorem 1.4.9 we have that $x \in \overline{A} = A$. So A is complete.

Theorem 2.3.4 The *p*-quasi cone metric space (X, q_p) is complete if and only if every sequence of closed sets $A_1 \supset A_2 \supset \ldots A_n \supset \ldots$, where $\delta(A_n) \to 0$ have an unique common point $x \in \bigcap_{n \in N} A_n$.

Proof: *Necessary condition.* $\forall n \in N$ we take $x_n \in A_n$. The sequence (x_n) is Cauchy in (X, q_p) due to $d_p(x_n, x_m) << \delta(A_n) \to 0$. Since X is complete then $x_n \to x \in X$. For $\forall k \in N$ fixed and $\forall n \in N, x_{k+n} \in A_{k+n} \subset A_k$. So (x_{k+n}) is a sequence in A_k. Since $(x_{k+n})_{k \in N}$ is a subsequence of (x_n) then $x_{k+n} \to x$ and x is a meeting point for A_k. Since A_k is closed $x \in \overline{A_k} = A_k$. So $\forall k \in N, x, y \in A_k \Rightarrow x \in \bigcap_{n \in N} A_n$.

Let prove now that x is unique. If there exist $y \in \bigcap_{n \in N} A_n$, then: $\forall n \in N, x, y \in A_n \Rightarrow d_p(x, y) \leq \delta(A_n) \to 0 \Rightarrow d_p(x, y) = 0 \Leftrightarrow x = y$.

Sufficient condition. Let (x_n) be a Cauchy sequence in (X, q_p). Sign as $A_n = \{x_n, x_{n+1}, \ldots\}$. Due to $\lim_{n,m} d_p(x_n, x_m) = 0 \Rightarrow \delta(A_n) \to 0$.

We have that $\delta(\overline{A_n}) \leq p^2 \delta(A_n)$. Taking limit of both sides we have $\delta(\overline{A_n}) \underset{n \to \infty}{\to} 0$ and $\overline{A_1} \supset \overline{A_2} \supset \ldots \overline{A_n} \supset \ldots$ Then from condition there exist x such that $x \in \bigcap_{n \in N} \overline{A_n}$. $\forall n \in N, x \in A_n$ and $d_p(x_n, x) \leq \delta(\overline{A_n})$. So $d_p(x_n, x) \to 0$, $x_n \to x$ and X is complete.

Definition 2.3.5 The *p*-quasi cone metric space (X, q_p) is compact if every open cover of *X* has a finite subcover.

$A \subset X$ is compact if every open cover of A has a finite subcover.

Definition 2.3.6 The *p*-quasi cone metric space (X, q_p) is sequentially compact if every $(x_n) \subset X$ has a convergent subsequence in X.

$A \subset X$ is sequentially compact if every $(x_n) \subset A$ has a convergent subsequence in A.

Theorem 2.3.7 Every sequentially compact *p*-quasi cone metric space (X, q_p) is complete.

Proof: The sequence (x_n) is Cauchy in $X \Rightarrow \lim_{n,m} d_p(x_n, x_m) = 0.$ Since X is sequentially compact, for sequence (x_n) there exist a subsequence $(x_{n_k}) \to x \in X$. So $d_p(x_{n_k}, x) \to 0.$ From inequality $d_p(x_n, x) << p(d_p(x_n, x_{n_k}) + d_p(x_{n_k}, x)) \to 0$ and (x_n) converges in X. So X is complete.

Theorem 2.3.8 Every sequentially compact set $A \subset X$, in (X, q_p) is closed and bounded.

Proof: Let be $x \in \overline{A}$ then $\exists (x_n) \subset A$ such that $x_n \to x.$ Since A is sequentially compact, for sequence $(x_n) \subset A$, exist subsequence (x_{n_k}) such that $x_{n_k} \to y \in A.$ As subsequence of (x_n) we have that $x_{n_k} \to x.$ From uniqueness of limit $x = y \in A.$ So $A = \overline{A}$ and A is closed. If A is unbounded, for $x_0 \in A$ and $c >> 0, \exists x_1 \in A$ such that $d_p(x_0, x_1) > p \cdot c$ where p is constant of *p*-quasi cone metric space. For the same reason $\exists x_2 \in A, d_p(x_2, x_0) > p(c + d_p(x_0, x_1)).$

In general for $n \in N$ there exist $x_n \in A$ such that

$$d_p(x_n, x_0) > p(d_p(x_1, x_0) + d_p(x_2, x_0) + ... + d_p(x_{n+1}, x_0) + c).$$

So for $n > m$ we have that $d_p(x_n, x_0) > p(d_p(x_m, x_0) + c).$ But

$d_p(x_n, x_0) \le p(d_p(x_n, x_m) + d_p(x_m, x_0),$

$p \cdot d_p(x_n, x_m) \ge d_p(x_n, x_0) - p \cdot d_p(x_m, x_0) \ge p \cdot c + p \cdot d_p(x_m, x_0) - p \cdot d_p(x_m, x_0)$ $p \cdot d_p(x_n, x_m) \ge p \cdot c$

or $d_p(x_n, x_m) \ge c >> 0, \forall n > m.$

So the sequence (x_n) has not any convergent subsequence, so A is not sequentially compact. This is a contradiction and A is bounded.

The inverse of theorem is not true because it is not true in metric space.

Theorem 2.3.9 Every closed set A in a sequentially compact *p*-quasi cone metric space (X, q_p) is sequentially compact.

Proof: Let (x_n) be a sequence in $A \subset X$. Since X is sequentially compact then the sequence (x_n) has a convergent subsequence (x_{n_k}) in $x \in X$. Due to (x_{n_k}) is a sequence in A and $x_{n_k} \to x \in X$, then x is a meeting point of A and since A is closed we have that $x \in A$ and A is sequentially convergent.

From Theorems 1.5.8 and 1.5.9 we have:

Theorem 2.3.10 The set $A \subset X$, where (X, q_p) is a sequentially compact *p*-quasi cone metric space is sequentially compact if and only if A is closed.

References:

- Abdeljawad. T, Karapinar. E. (2009) Quasicone Metric Spaces and Generalizations of Caristi Kirk's Theorem. *Fixed Point Theory and Applications*, 2009 :574387, doi:10.1155/2009/574387
- Cvetković. A, Stanić P. Marija, Dimitrijević. S and Simić. S., (2011) Common Fixed Point Theorems for Four Mappings on Cone Metric Type Space. *Fixed point Theory and Applications*, Vol 2011, Article ID :589725
- Deimling. K, (1985) *Nonlinear Functional Analysis*, Springer, Berlin, Germany.
- Huang. L, Zhang. X, Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl. 332*, 1468 – 1476.
- Proinov.P. (2013) A unified theory of cone metric spaces and its applications to the fixed point theory. *Fixed point Theory and Applications*, 2013 :103
- Rezapour. Sh, Derafshpour.M, Hamlbarani R., (2008). A Review on Topological Properties of Cone Metric Space. In *Analysis Topology and Aplications* 2008 (ATA 2008) Teknical Faculty Coca, University of Kragujevac,Vrujacka, Serbia, 30 May - 4 June
- Sila. E, Hoxha. E, Dule. K, (2014). Some fixed point results on *p*-Quasi-Cone Metric Space, *Proceedings in Advanced Research in Scientific Areas*, pp 277-282.